Int. J. Heat Mass Transfer. Vol. 7, pp. 1395–1404. Pergamon Press 1964. Printed in Great Britain

THE INFLUENCE OF A WAVY, MOVING INTERFACE ON PRESSURE DROP FOR FLOW IN CONDUITS

F. W. CHANG and A. E. DUKLER
University of Houston, Houston, Texas

(*Received* 18 *June* 1964)

Abstract—Approximate theoretical solutions have been obtained for laminar flow in a conduit with wavy moving boundaries. The results predict the data of Laird quite well and demonstrate that the oscillation of the boundary contributes to the apparent increased friction. This suggests that studies of the interface in two phase flow concern themselves not only with the wave profiles but with the wave frequency as well.

The effect of a moving boundary in turbulent flow is shown to depend on the relative magnitude of the random fluctuation due to turbulence and the periodic fluctuation induced by the moving boundary.

Experimental equipment is described in which turbulent intensity measurements were made across a wall oscillating with a standing wave having a length of 2 in and an amplitude of +0·025. Even with this large wavelength and low amplitude, the oscillating wall caused increased apparent turbulent shear stress.

Because analytical solution of the equations of motion are not possible, an extension of the phenomenological method to the case of moving walls is outlined.

NOMENCLATURE

A, R^{-n};
a_o, maximum deviation of radius from R (see Fig. 7);
C, numerical constant in equation (18a);
F, frequency;
F_R, energy dissipated due to friction for flow across a rough surface;
F_S, energy dissipated due to friction for flow across a smooth surface;
f_m, friction factor for moving wall;
f_s, friction factor for stationary wall;
g_c, constant in Newton's second law;
g_L, acceleration of gravity;
h, coordinate in direction of gravity;
k, wave number $= \frac{2\pi}{\lambda}$;
m, numerical constant in equation (18a);
n, damping constant [see equation (10)];
$\bar{p}$, time average local pressure;
R, radius on non-oscillating wall (see Fig. 7);
Re, Reynolds number;
r, radial coordinate;
t, time,
u, v, w, instantaneous velocity in coordinate directions;
$\bar{u}, \bar{v}, \bar{w}$, time average velocities in coordinate directions;
$\bar{u}_w, \bar{v}_w$, time average velocities at the wall;
u', v', w', fluctuating velocities in coordinate directions;
u^+, dimensionless velocity parameter, $\bar{u}/(\tau_o g_c/p)^{1/2}$;
$\bar{u}_{avg}$, time average velocity, averaged over the flow area;
v'_w, fluctuating velocity at wall;
W_A, work required to form surface area;
x, axial coordinate;
y^+, dimensionless distance parameter, $\left(\frac{\tau_o g_c}{\rho}\right)^{1/2} \frac{y}{\nu}$.

Greek symbols
ϵ, numerical constant in equation (18b);
ϵ_R, eddy viscosity due to random turbulent fluctuations;
ϵ_p, eddy viscosity due to periodic fluctuations induced by wall motion;
λ, wavelength;
ν, kinematic viscosity;
ρ, density;
τ_o, wall shear stress;
ω, angular frequency $= 2\pi F$.

INTRODUCTION

THE SIMULTANEOUS flow of gas and liquid in a conduit is accompanied by rates of momentum transfer (and frictional pressure drop) which are considerably higher than experienced when gas flows alone. In recent years considerable effort has been devoted to explaining these higher transport rates. A few of the many examples can be found in references [1–3]. A survey of these approaches appears in reference [4]. The essence of the problem is to find a model describing the mechanism which is physically realistic and which is not so complex as to render the mathematics intractable.

SOME POSSIBLE MECHANISMS

Two phase flow is characterized by the presence of an interface. Except for bubble flow at low gas rates and completely dispersed liquid flow at very high gas rates, these interfaces are of substantial continuous area and are covered by waves. Figure 1 shows the surface of a liquid film in the presence of a very low countercurrent gas flow rate relative to the liquid surface. This surface is tremendously complex and experimental observations [5, 6], reveal that with strong concurrent gas flows the complexity increases. Now we need to determine the mechanism(s) by which the interface makes its influence known to the gas phase. Several possibilities exist, some or all of which may act simultaneously. It is of interest to examine a number of these, looking at each as if it was acting alone.

The gas does work on the interface to form surface

Figure 2 demonstrates this mechanism. If the mechanical energy balance is applied to the gas flowing across a solid and liquid surface, respectively, the two equations shown in Fig. 2 are obtained. For equivalent conditions of kinetic and potential energy changes the measured pressure drop for flow across the fluid interface will be larger by the amount of work done by the gas to maintain the wave motion, even if it is assumed that the frictional dissipation across the liquid and the solid is the same. There is at present no way to separate the work and friction terms experimentally; thus, when experimental pressure drop is used to calculate friction, a higher apparent friction loss is obtained. Should this be an important mechanism, then an understanding of the higher momentum losses must come about by experimental and analytical studies of interfacial surface renewal rates and the way in which flow rate, fluid properties and conduit geometry influence these rates.

Solid: $$-\frac{\Delta\bar{p}}{\rho} = \frac{\Delta\bar{v}^2}{2g_c} + \frac{g_L}{g_c}\Delta h + F$$

Fluid: $$-\frac{\Delta\bar{p}}{\rho} = \frac{\Delta\bar{v}^2}{2g_c} + \frac{g_L}{g_c}\Delta h + F + W_A$$

$$W_A = \sigma A$$

FIG. 2. Mechanical energy balance for flow across solid and fluid interfaces.

If one uses experimental pressure drop data for two phase flow and at equivalent gas rates calculates the single phase pressure drop, it is possible to estimate the magnitude of the work term and explore its reasonableness. Such calculations show that, in order to account for the higher pressure drop by a work mechanism, it would be necessary that the interfacial surface be renewed 500 to 1000 times per second. an unreasonably high rate. Thus it is necessary to conclude that the energy required to form surface cannot possibly make a major contribution to the apparent increased friction in two phase flow.

The liquid interface appears as a rough solid wall to the moving gas

This mechanism is demonstrated in Fig. 3. Assume that the liquid could be solidified in such a way that the interfacial solid roughness is identical in shape to the wave profile on the surface of the liquid. Since it is known that flow across rough surfaces is accompanied by higher frictional energy loss than flow across smooth ones, the measured pressure drops (as given by the mechanical energy balance in Fig. 3) will be higher. Should this model be a valid one, then it is necessary to concentrate on a study of the

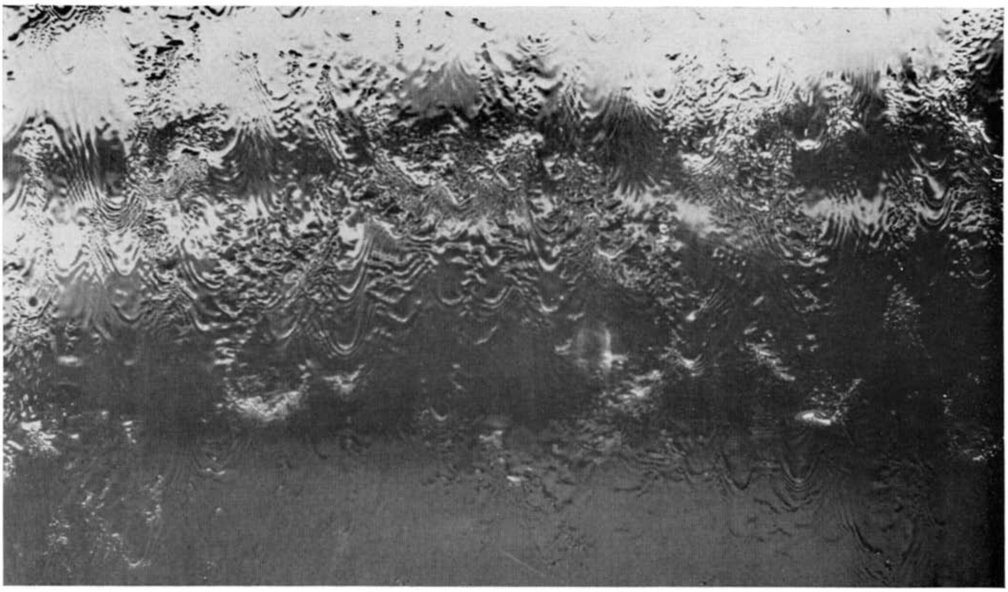


FIG. 1. Wave motion on a vertical film surface.

$$\text{Solid} = \frac{\Delta\bar{p}}{\rho} = \frac{\Delta\bar{v}^2}{2g_c} + \frac{g_L}{g_c}\Delta h + F_s$$

$$\text{Fluid} = \frac{\Delta\bar{p}}{\rho} = \frac{\Delta\bar{v}^2}{2g_c} + \frac{g_L}{g_c}\Delta h + F_R$$

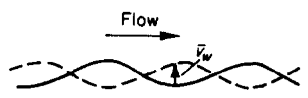
FIG. 3. Mechanical energy balance for flow across solid smooth and solid rough interfaces.

interfacial profile. Then this profile can be duplicated in solid surfaces and the influence of the roughness on increased friction studied in single phase experiments, a much simpler experimental medium.

The form drag due to the waves undoubtedly contributes significantly to the increased pressure drop. However, several studies [7, 8], have attempted to characterize the interface by an equivalent roughness. While the results of any one experiment can be used to calculate an equivalent roughness for that condition, it has not been possible to generalize this approach to predict results at other conditions where the interfacial roughness appeared essentially the same. This suggests that the roughness heights alone are not sufficient.

Waves on the interface cause velocity fluctuations in the gas phase normal to flow which change the shear stress

Figure 4 illustrates this mechanism. The equation shown is the two dimensional form of the Navier Stokes equation. (While this equation for laminar flow is included for illustrative purposes, the reasoning applies as well to turbulent flow, as will be shown below.) When the gas flows across a non-moving solid boundary (the horizontal solid line), at the boundary the time average velocity in the direction of flow at the wall, $\bar{u}_w$, and the time average velocity normal to the mean flow at the wall, $\bar{v}_w$, are both zero. Now consider a wavy fluid interface. The solid wavy line represents the fluid interface at one instant of time and the dotted wavy line is the same interface an instant later after the wave has travelled along the surface. For this case the velocities, $\bar{u}_w$ and $\bar{v}_w$, are now not zero and depend on the degree of oscillation of the interface. One can expect the equations of motion to predict a different value for pressure drop when these boundary conditions are not zero (two phase flow) than when they are zero (single phase flow).



$$\bar{u}\frac{\delta\bar{u}}{\delta x} + v\frac{\delta\bar{u}}{\delta x} = -\frac{\delta\bar{P}}{\delta x} + \frac{\mu}{\rho}\left[\frac{\delta^2\bar{u}}{\delta x^2} + \frac{\delta^2\bar{u}}{\delta y^2}\right]$$

Solid	Fluid
$y = 0 \quad \bar{u}_w = 0$	$y = 0 \quad \bar{u}_w \neq 0$
$\bar{v}_w = 0$	$\bar{v}_w \neq 0$

FIG. 4. Boundary conditions in the equation of motion for flow across smooth and wavy moving surfaces.

There is interesting experimental evidence that this is a significant mechanism. In 1954, Laird [9] constructed an apparatus in which he pumped water in laminar flow through a flexible tube and imposed oscillations on the wall at a number of frequencies, amplitudes and wavelengths. A schematic diagram of the Laird equipment is shown in Fig. 5. The wall was oscillated between positions shown as solid and

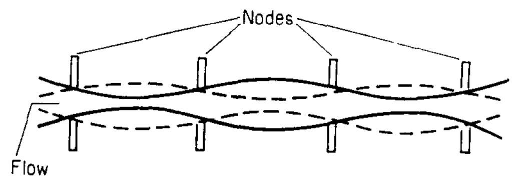


FIG. 5. Laird's oscillating tube wall.

dotted, respectively, while liquid flowed as shown. Typical experimental results are shown in Fig. 6. It is clear that the oscillation of the wall had a marked effect on the friction. Laird presented no theoretical analysis for these observations. This reasoning and Laird's data suggests that before we can understand the effect of the interface on pressure drop we must study not only the profile but the oscillation of the interface as well.

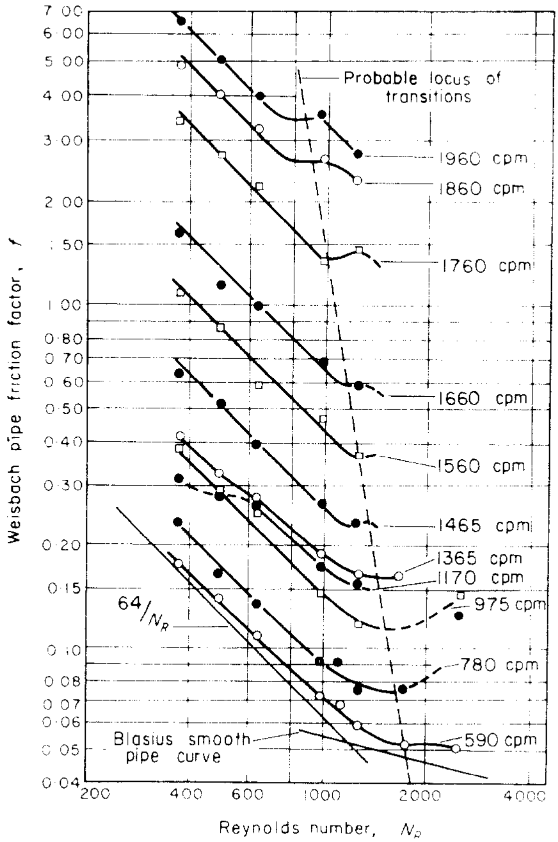


FIG. 6. Typical experimental results of Laird.

OBJECTIVES OF THIS STUDY

In this paper we will present an analysis which explains the Laird data, thus demonstrating that, at least in laminar flow, mechanism *c*, discussed above is important. This analysis is then extended to turbulent flow and the factors which must be determined experimentally to complete the understanding are evolved. The experimental equipment designed and constructed to explore this effect for turbulent flow is discussed and preliminary experimental results presented.

LAMINAR FLOW IN A TUBE WITH WALLS OSCILLATING AS AXISYMMETRIC STANDING WAVES

Assume that, due to the oscillation of the wall, fluctuating velocity components are induced in the axial and radial directions. The instantaneous velocity components of the flow, u, v and w, can be expressed as

$$\begin{aligned} u &= \bar{u} + u' \\ v &= \bar{v} + v' \\ w &= \bar{w} + w' \end{aligned} \tag{1}$$

$\bar{u}$, $\bar{v}$, $\bar{w}$ are mean values, and u', v', w' are the oscillating components. In the equations which follow, x is the axial and r the radial coordinate.

We assume that $w = 0$ and that the oscillation of the wall does not induce fluctuation in the circumferential direction so that $w' = 0$. Substituting equation (1) into the Navier-Stokes equations in cylindrical coordinates and taking time averages gives:

$$\left.\begin{aligned} &\frac{1}{r}\frac{\partial(r\bar{u}\bar{v})}{\partial r} + \frac{\partial \bar{u}'^2}{\partial x} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\left[\frac{\partial^2 \bar{u}}{\partial r^2} + \right. \\ &\left.\frac{1}{r}\frac{\partial \bar{u}}{\partial r} + \frac{\partial^2 \bar{u}}{\partial x^2}\right] - \frac{1}{r}\frac{\partial(r\overline{u'v'})}{\partial r} - \frac{\partial^2(\overline{u'^2})}{\partial x^2} \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned} &\frac{1}{r}\frac{\partial(r\bar{v}^2)}{\partial r} + \frac{\partial(\bar{u}\bar{v})}{\partial x} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r} + \nu\left[\frac{\partial^2 \bar{v}}{\partial r^2} + \right. \\ &\left.\frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 v}{\partial x^2} - \frac{\bar{v}^2}{r^2}\right] - \frac{1}{r}\frac{\partial(r\overline{v'^2})}{\partial r} - \frac{\partial(\overline{v'u'})}{\partial x} \end{aligned}\right\} \tag{3}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{1}{r}\frac{\partial(r\bar{v})}{\partial r} = 0 \tag{4}$$

From (1) and (4) we also get

$$\frac{\partial u'}{\partial x} + \frac{1}{r}\frac{\partial(rv')}{\partial r} = 0 \tag{5}$$

If we consider the case where $\bar{v}$ is small, then from the equation of continuity $\partial\bar{u}/\partial x$ is also small and $\partial^2\bar{u}/\partial x^2$, $\partial(r\bar{u}\bar{v})/\partial r$ are negligible compared to $\partial^2\bar{u}/\partial r^2$. Taking all this into account, equations (3) and (4) can be simplified to:

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \nu\left(\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{u}}{\partial r}\right) - \frac{1}{r}\frac{\partial(r\overline{u'v'})}{\partial r} - \frac{\partial(\overline{u'^2})}{\partial x^2} \tag{6}$$

$$0 = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r} - \frac{1}{r}\left(\frac{\partial r\overline{v'^2}}{\partial r}\right) - \frac{\partial \overline{u'v'}}{\partial x} \tag{7}$$

In these equations, there exist more unknowns. i.e. $\bar{p}$, $\bar{u}$, $\overline{v'^2}$, $\overline{u, v}$, than are equations; i.e. [5, 6, 7]. It is necessary to seek further relations in order to solve the problem. We accomplish this by first finding an expression for the velocity fluctuation at the oscillating boundary, v_w', and then suggesting a manner in which this is damped as it progresses into the fluid.

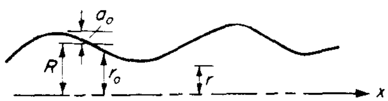


FIG. 7. Coordinates of the wave.

Consider the system as shown in Fig. 7. The oscillating wall of the tube can be described by

$$r_o = R + a_o \cos kx \sin \omega t \tag{8}$$

At the wall the component of fluctuating velocity in the radial direction can be obtained by differentiating equation (8).

$$v_w' = \frac{\partial r_o}{\partial t} = a_o\, \omega \cos kx \cos \omega t \tag{9}$$

It is likely that this displacement damps in some manner as the disturbance approaches the center of the tube. The damping equation assumed is given by (10).

$$v' = \left(\frac{r}{R}\right)^n v_w' = r^n A_o\, \omega \cos kx \cos \omega t \tag{10}$$

where $A = R^{-n}$. With this equation it is possible to find the product rv' and its r derivative and so, from equation (5), obtain an equation for local velocity fluctuation, u', as a function of the two position variables and time.

$$u' = -(n+1)\frac{A}{k} r^{n-1}\, \omega a_o \sin kx \cos \omega t + C_1(r) \tag{11}$$

The constant of integration, $C_1(r)$ can be found from the condition, at $x = 0$, π/k, $2\pi/k, \ldots$ $u' = 0$, therefore $C_1(r) = 0$

$$u' = -(n+1)\frac{A}{k} r^{n-1}\, \omega a_o \sin kx \cos \omega t \tag{12}$$

From these equations the time average of the two fluctuating terms which appear in equation (6) can be calculated:

$$\frac{\partial(\overline{ru'v'})}{\partial r} = -\frac{(n+1)}{2k} A^2\, \omega^2\, a_0^2\, r^{2n-1} \sin 2kx \tag{13}$$

$$\frac{\partial \overline{u'^2}}{\partial x} = \frac{(n+1)^2}{2k} A^2\, \omega^2\, a_o^2 \sin(2kx)\, r^{2n-2} \tag{14}$$

Substituting equations (13) and (14) into equation (6) gives, after integration with respect to r:

$$\bar{u} = -\frac{1}{4\mu}\frac{\partial \bar{p}}{\partial x}(R^2 - r^2) - \frac{(n+1)}{8k\nu n^2} A^2\, \omega^2\, a_o^2 (R^{2n} - r^{2n}) \sin 2kx \tag{15}$$

For this integration suitable boundary conditions are: at $r = 0$, $\mathrm{d}\bar{u}/\mathrm{d}r = 0$ for all values of x and at $r = R$, $\bar{u} = 0$ for all values of x.

Equation (15) then gives the distribution of time average velocity radially. The average velocity over the tube and the pressure gradient related to this average is:

$$\bar{u}_{\text{avg}} = \frac{1}{\pi R^2}\int_0^R \bar{u}\, 2\pi\, r\, \mathrm{d}r = -\frac{R^2}{8\mu}\frac{\partial \bar{p}}{\partial x} - \frac{A^2\, \omega^2\, a_o^2}{8nk\nu} R^{2n} \sin 2kx$$

The maximum value of $-\mathrm{d}\bar{p}/\mathrm{d}x$ is:

$$-\left(\frac{\partial \bar{p}}{\partial x}\right)_m = \frac{8\mu\ \bar{u}_{\text{avg}}}{R^2}\left(1 + \frac{\omega^2\, a_o^2}{8\nu kn\, \bar{u}_{\text{avg}}}\right) \tag{16}$$

For stationary smooth walls

$$-\left(\frac{\mathrm{d}p}{\partial x}\right)_s = \frac{8\mu\, \bar{u}_{\text{avg}}}{R^2}$$

Therefore the ratio of the friction factors of the moving to the smooth wall becomes:

$$\frac{f_m}{f_s} = \frac{-(\partial p/\partial x)_m}{-(\partial p/\partial x)_s} = 1 + \frac{\omega^2\, a_o^2}{8\nu kn\, \bar{u}_{\text{avg}}} = 1 + \frac{\pi\, F^2\, a_o^2\, \lambda\, R}{2\, \nu^2\, n\, Re} \tag{17}$$

Equation (17) would enable us to calculate the friction factor for laminar flow inside oscillating tubes if n was known.

From equation (10) it is seen that n is a quantity describing the damping of the fluctuation induced by the wall motion. As n increases, the wall fluctuation is suppressed more rapidly. Therefore n is smaller for larger amplitude and larger frequency. Also n would be greater for larger kinematic viscosity or smaller Reynolds number. The calculation of n for a number of conditions of Laird's data shows that these qualitative observations are valid and the principal variables which determine n are:

$$n = \phi(Re, F, a_o, R, \lambda)$$

It is possible to form a number of characteristic dimensionless groups which include the variables. These include

$$\frac{Fa_o^2}{\nu}, \frac{\lambda}{a_o}, \frac{R}{a_o}, \frac{Fa_oR}{\nu}, \frac{FR^2}{\nu}, Re, \frac{\lambda}{R}$$

A study of these groups in relation to the Laird data suggests they combine in the form

$$n = \frac{C}{Re}\left(\frac{\nu}{F\,a_o\,R}\right)\left(\frac{\lambda}{R}\right)\exp\left[-m\left(\frac{F\,a_o\,R}{\nu}\right)^2\right] \tag{18a}$$

This gives, when substituted into equation (17)

$$\frac{f_m}{f_s} = 1 + \epsilon\left(\frac{F\,a_o\,R}{\nu}\right)^3\exp\left[m\left(\frac{F\,a_o\,R}{\nu}\right)^2\right] \tag{18b}$$

Equation (18b) shows that the friction factors for moving and stationary boundaries are related through the frequency, displacement amplitude and non-oscillating radius, combined in a dimensionless oscillation Reynolds number. It is of interest to note that, expressed in log form:

$$\log f_m = \log\left\{64 + 64\,\epsilon\left(\frac{F\,a_o\,R}{\nu}\right)^3 \exp\left[m\,\frac{Fa_o\,R}{\nu}\right]\right\} - \log Re$$

this equation predicts a log linear friction factor Reynolds number with a slope of minus one. Thus the f_m vs. Re curves should be parallel to and displaced from the single phase curve, exactly as seen in the Laird data, Fig. 6.

When the Laird data at minimum displacement value, $a_o/R = 0{\cdot}043$, are used to evaluate the constants ϵ and m and then equation (18b) is used to predict the results for all of the Laird data, the results shown in Fig. 8 are obtained. The solid lines represent the prediction of the theoretical equation. The points represent the experimental data. Agreement is quite satisfactory.

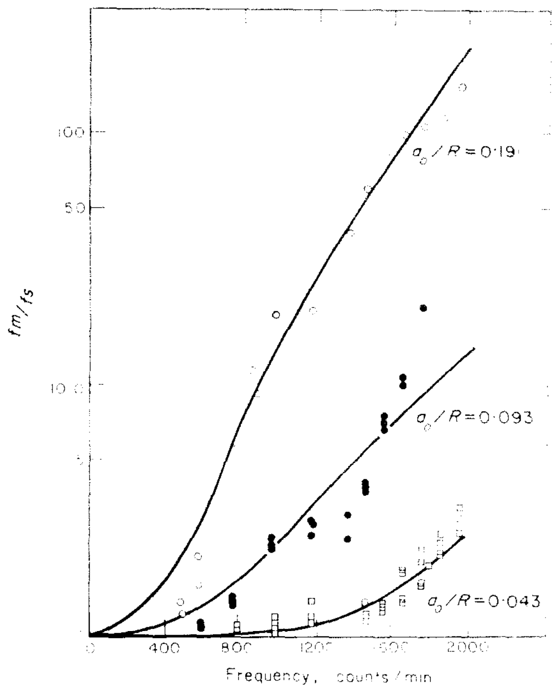


FIG. 8. Comparison of theory—(solid lines) with data of Laird—(points).

TURBULENT FLOW ACROSS A WAVY MOVING BOUNDARY

Theory

The mechanism of momentum transfer and friction for turbulent flow across plane, non-moving boundaries has, as yet, not yielded to rigorous analysis. Because of the impossibility of solving the Reynolds equations without added independent equations to describe the turbulent shear stress, the most fruitful approaches have been phenomenological ones. Recently, Konobeev and Zhavaronov [10] applied simplified turbulent models to flow across stationary rough surfaces with good success. It is apparent that the case of a moving boundary would be of significantly increased complexity.

One way to view the behavior of the fluid is to assume that the instantaneous velocity can be expressed in terms of a time average value, a random fluctuation due to turbulence and a periodic fluctuation due to the wall oscillation.

$$u = \bar{u} + u' + u''$$
$$v = \bar{v} + v' + v''$$
$$w = \bar{w} + w' + w'' \qquad (19)$$

If these definitions are substituted into the Navier–Stokes equations (following Reynolds) and time averages taken, then equations for the oscillating wall case can be developed. For steady conditions and negligible acceleration, the x direction equation becomes:

$$0 = -\frac{\partial \bar{p}}{\partial x} + \mu \nabla^2 \bar{u} - \rho \left[\frac{\partial \overline{u'^2}}{\partial x'} + \frac{\partial(\overline{u'v'})}{\partial y} + \frac{\partial(\overline{u'w'})}{\partial z}\right] - \rho \left[\frac{\partial \overline{u''^2}}{\partial x} + \frac{\partial \overline{u''v''}}{\partial y} + \frac{\partial(\overline{u''w''})}{\partial z}\right] \qquad (20)$$

In arriving at equation (20) it was assumed that there is zero correlation between any of the random and periodic terms, an assumption that seems completely justified in view of the completely different nature of the two fluctuations.

In the three dimensional case, then, each equation has three stress terms due to the periodic component in addition to the usual three turbulent stress terms. While it is clear that no serious attempt need be made to solve these questions, it can be seen that the movement of the wall is important only to the extent that this motion induces periodic fluctuations which are of the order of (or larger than) the random fluctuations.

Experimental equipment

In order to make a direct measurement of relative magnitude of the periodic and turbulent fluctuations, an oscillating wall channel shown schematically in Fig. 9 was constructed. This channel has the dimensions of 18 in by 1·5 in by 5·5 ft. The top and three sides were made of clear plastic and the bottom surface was formed

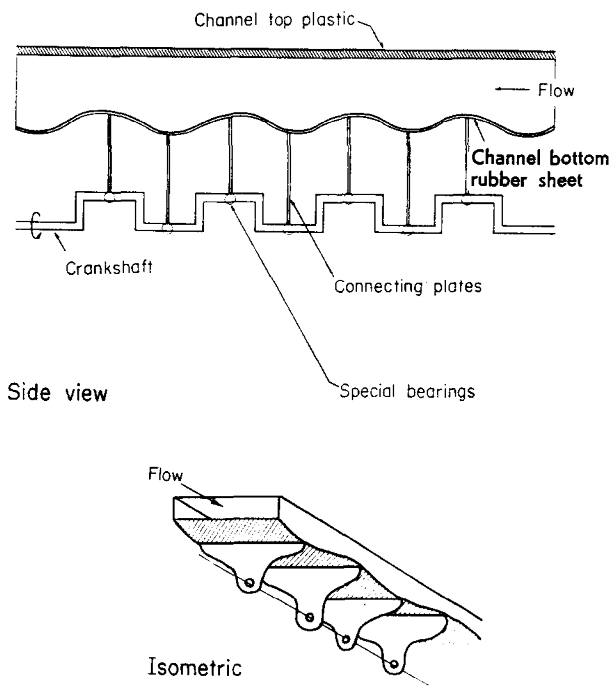


FIG. 9. Schematic views of the oscillating wall channel.

by a rubber sheet. To the bottom surface 52 aluminum plates were glued 1 in apart. These connecting plates were attached to a connecting rod using specially constructed bearings. The camshaft was driven by a $\frac{3}{4}$ h.p. motor through a pulley system and an electrical variable speed drive. The camshaft was especially fabricated to give a maximum wave amplitude of 0·025 in. This system thus generated a standing wave with a wavelength of 2 in, a wave amplitude of 0·025 in and an oscillation frequency varying from zero (no cam rotation) to 1200 rev/min. Air was provided by a rotary compressor and after being filtered and metered, flowed into a 6 ft entrance section, identical in all respects to the test section except that the rubber surface was replaced by a solid plastic sheet similar to the other walls. Turbulence measurements were made with a Flow Corporation constant current hot wire anemometer, Model HWB-2. Readings of turbulence intensity and cross correlations were made on a true random signal voltmeter designed for 16 s averaging of the signal. Additional details of the equipment, the measuring technique and preliminary data appear in reference 11.

Typical experimental results

Figure 10 shows the measured relative turbulent intensities for a stationary wall with no fixed deflection and for a wall oscillating at 800 cycles/s. The effect is small but distinct and reproducible. For these experiments the gas Reynolds number was approximately 20 000. Equipment modifications are now under way to permit measurements for smaller wavelengths and larger amplitudes, conditions more nearly representing liquid surfaces in two phase flow.

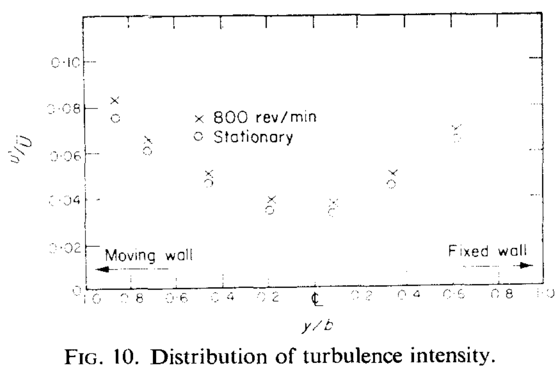


FIG. 10. Distribution of turbulence intensity.

It thus appears that the movement of the wall induces added stress in the turbulent stream which is of the order of the random turbulent stresses themselves. In the experiments reported here this periodic contribution was small. However, as wavelength of the surface is decreased and wave amplitude increased, these effects can be expected to be more significant.

A phenomenological approach

The problem of solving the Reynolds equations can be approached phenomenologically by assuming the existence of an eddy diffusion coefficient for the periodic contribution to shear similar to that used for the random fluctuation. The one dimensional momentum equation can then be written as

$$0 = -\frac{\partial \bar{p}}{\partial x} + \mu \frac{\partial^2 \bar{u}}{\partial y^2} + \rho \frac{\partial}{\partial y}\left[(\epsilon_R + \epsilon_p)\frac{\partial \bar{u}}{\partial y}\right] \quad (21)$$

Here ϵ_R is the random contribution to the eddy viscosity, while ϵ_p is the periodic term. It seems reasonable to assume that the turbulent term is unaffected by the wall motion (there being no correlation between random and periodic modes) and thus it should be possible to use existing relationships that have proved satisfactory. For the region near the wall, Deissler [12] has suggested a useful equation

$$\epsilon_R = n^2 \bar{u}\, y$$

where n is a constant determined from experiment.

In the presence of wall oscillations the total eddy viscosity, ϵ_T, could be written as:

$$\epsilon_T = n^2 \bar{u}\, y + \epsilon_p$$

If it is further assumed that the effects of the oscillations are most significant near the wall and that the major effect is to displace the velocity distribution curve due to its effect at the boundary, ϵ_p can be considered constant and a modified Deissler expression for velocity distribution near the wall can be obtained.

$$y^+ = [\exp (nu^+/\sqrt{2})] \left[\frac{1}{n} \sqrt{\left(\frac{\pi}{2}\right)} \left(1 + \frac{\epsilon_p}{\nu}\right) \operatorname{erf}\left(\frac{nu^+}{\sqrt{2}}\right)\right] \quad (22)$$

Now it remains to determine from experimental data the term, ϵ_p, and how the wall motion influences the magnitude of this quantity. Experimental measurements of this type are now underway.

ACKNOWLEDGEMENT

This work was supported, in part, by the American Gas Association and the American Petroleum Institute under a joint research grant for studies in Two Phase Flow.

REFERENCES

1. R. W. LOCKHART and R. C. MARTINELLI, *Chem. Engng Progr.* **45**, 39 (1949).
2. J. M. CHENOWETH and M. W. MARTIN, *Petrol Ref* **34**, 151 (1955).
3. C. J. HOOGENDOORN, *Chem. Engng Sci.* **9**, 205 (1959).
4. A. E. DUKLER and M. WICKS, *Modern Chemical Engineering*, Vol. I, chapt. 8. Reinhold, New York (1963).
5. T. J. HANRATTY and J. M. ENGIN, *J. Amer. Inst. Chem. Engrs* **3**, 299 (1957).
6. N. H. TAYLOR and G. F. HEWITT, *AERE* Report 39-52 (1962).
7. L. O. LILLELEHT and T. J. HANRATTY, *J. Amer. Inst. Chem. Engrs* **1**, 548 (1961).
8. J. R. WROBEL and H. N. MCMANUS, *Develop. Mech.* **1**, 578 (1961).
9. A. D. K. LAIRD, *Trans. Amer. Soc. Mech. Engrs* **76**, 1005 (1954).
10. V. I. KONOBEEV and N. M. ZHAVORONKOV, *Int. Chem. Engng* **2**, 431 (1962).
11. FU-WU CHANG, M.S. Thesis, University of Houston (1964).
12. R. G. DEISSLER, *NACA TN* 2138 (1950).

Résumé—Des solutions théoriques approchées ont été obenues pour l'écoulement laminaire dans une conduite avec des parois en mouvement ondulatoire. Les résultats prévoient très bien les résultats expérimentaux de Laird et démontrent que l'oscillation de la paroi contribue à l'augmentation apparente du frottement. Ceci suggère que les études de l'interface dans l'écoulement diphasique sont concernées non seulement par les profils d'onde mais également par la fréquence de l'onde.

On montre que l'effet d'une frontière mobile en écoulement turbulent dépend de la grandeur relative de la fluctuation aléatoire due à la turbulence et de la fluctuation périodique provoquée par la frontière mobile.

On décrit l'équipement expérimental avec lequel les mesures d'intensité de turbulence ont été faites de long d'une paroi oscillante avec une onde stationnaire de longueur d'onde égale à 5 cm et d'amplitude égale à 0,625 mm. Même avec cette grande longueur d'onde et cette faible amplitude, la paroi oscillante causait une augmentation de la contrainte de cisaillement turbulente apparente.

Parce que la solution analytique des équations du mouvement n'est pas possible, on a ébauché une extension de la méthode phénoménologique au cas des parois mobiles.

Zusammenfassung—Für laminare Strömung mit einer sich fortbewegenden Grenzlinie in einer Rohrleitung erhielt man theoretische Näherungslösungen. Die Ergebnisse bestätigen die Werte von Laird sehr gut und zeigen, dass die Schwingungen der Grenzlinie zur offensichtlich erhöhten Reibung beitragen. Dies legt nahe, dass sich Studien an einer Trennfläche in einer Zweiphasenströmung nicht nur mit dem Profil der Welle, sondern auch mit ihrer Frequenz zu befassen haben.

Die Wirkung einer sich bewegenden Grenzlinie in turbulenter Strömung hängt, wie gezeigt wird, von der relativen Grösse der zufälligen Änderung in einem Masse ab, das der Turbulenz und der periodischen Schwankung, hervorgerufen von der sich bewegenden Grenzlinie, entspricht.

Die experimentelle Vorrichtung, in der Messungen der Turbulenzstärke längs einer Wand ausgeführt werden, die wie eine stehende Welle von 5,08 cm Länge und +0,0635 cm Amplitude schwingt, wird beschrieben. Sogar bei dieser grossen Wellenlänge under der niedrigen Amplitude verursachte die schwingende Wand eine offensichlich turbulente, erhöhte Scherbeanspruchung. Weil eine analytische Lösung der Bewegungsgleichungen nicht möglich ist, wird eine Erweiterung der phänomenologischen Methode für den Fall sich bewegender Wände umrissen.

Аннотация—Получены приближенные теоретические решения уравнений для ламинарного потока в турбе с волнистыми подвижными стенками. Результаты решений подтвердили данные Леарда и показали, что колебание стенки способствует очевидному увеличению сопротивления. Поэтому исследования границы раздела в двухфазном потоке охватывают не только волновые профили, но и волновую частоту.

Показано, что влияние подвижной стенки в турбулентном потоке зависит от относительной величины произвольной флуктуации, вызванной турбулентностью, и периодической флуктуации, генерируемой подвижной стенкой.

Описывается экспериментальная установка, в которой измерялась интенсивность турбулентности на стенке, колебание которой вызывает стоячую волну длиной 2 дюйма и амплитудой +0,025. Даже при такой большой длине и малой амплитуде колеблющаяся стенка вызывает очевидное увеличение турбулентного касательного напряжения.

Поскольку аналитическое решение этих уравнений не возможно, обращается внимание на использование феноменологического метода для случая подвижных стенок.